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Discrete-time analogues of integrodifferential equations modelling bidirectional neural networks

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Abstract

We formulate discrete-time analogues of integrodifferential equations modelling bidirectional neural networks studied by Gopalsamy and He. The discrete-time analogues are considered to be numerical discretizations of the continuous-time networks and we study their dynamical characteristics. It is shown that the discrete-time analogues preserve the equilibria of the continuous-time networks. By constructing a Lyapunov-type sequence, we obtain easily verifiable sufficient conditions under which every solution of the discrete-time analogue converges exponentially to the unique equilibrium. The sufficient conditions are identical to those obtained by Gopalsamy and He for the uniqueness and global asymptotic stability of the equilibrium of the continuous-time network. By constructing discrete-time versions of Halanay-type inequalities, we obtain another set of easily verifiable sufficient conditions for the global exponential stability of the unique equilibrium of the discrete-time analogue. The latter sufficient conditions have not been obtained in the literature of continuous-time bidirectional neural networks. Several computer simulations are provided to illustrate the advantages of our discrete-time analogue in numerically simulating the continuous-time network with distributed delays over finite intervals. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Gopalsamy and He [4] studied the global asymptotic stability of equilibria of continuous-time bidirectional neural networks described by integrodifferential equations of the form

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} S\left(\int_0^\infty K_{ij}(s) y_j(t-s)\,\mathrm{d}s\right) + I_i,\\ \frac{\mathrm{d}y_i(t)}{\mathrm{d}t} &= -c_i y_i(t) + \sum_{j=1}^{m} d_{ij} S\left(\int_0^\infty H_{ij}(s) x_j(t-s)\,\mathrm{d}s\right) + J_i\end{aligned} \qquad (1.1)$$

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for $i \in \mathscr{I} = \{1, 2, \ldots, m\}$, $t > 0$ with initial values given by

$$x_i(s) = \varphi_i(s), \qquad y_i(s) = \psi_i(s) \quad \text{for } i \in \mathscr{I}, \; s \in (-\infty, 0], \tag{1.2}$$

where $\varphi_i(s)$, $\psi_i(s)$ denote real-valued continuous functions defined for $s \in (-\infty, 0]$ and

$$\sup_{s\in(-\infty,0]} |\varphi_i(s)| < \infty, \qquad \sup_{s\in(-\infty,0]} |\psi_i(s)| < \infty, \quad i \in \mathscr{I}.$$

The equilibria of the network (1.1) are referred to as patterns or memories associated with the exogeneous inputs I_i, J_i. System (1.1) denotes one of the possible modifications of bidirectional neural networks proposed and studied by Kosko [7–9]. In (1.1), the signal response function $S(\cdot)$ is bounded, nondecreasing, monotonic and differentiable, typically, $S(u) = \tanh(u)$ for $u \in \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers; the dissipative coefficients a_i, c_i, the synaptic weights b_{ij}, d_{ij} and the exogeneous inputs I_i, J_i are assumed to satisfy

$$a_i, c_i \in (0, \infty), \qquad b_{ij}, d_{ij}, I_i, J_i \in \mathbb{R}, \quad i, j \in \mathscr{I}; \tag{1.3}$$

the delay kernels $K_{ij}(s)$, $H_{ij}(s)$, $i, j \in \mathscr{I}$ defined for $s \in [0, \infty)$ denote nonnegative real-valued continuous functions satisfying

$$\begin{aligned} &\int_0^\infty K_{ij}(s) = 1, \qquad \int_0^\infty sK_{ij}(s)\,\mathrm{d}s < \infty, \\ &\int_0^\infty H_{ij}(s) = 1, \qquad \int_0^\infty sH_{ij}(s)\,\mathrm{d}s < \infty. \end{aligned} \tag{1.4}$$

We note that a class of delay kernels of the above type is given by

$$K(s) = \frac{\gamma^{r+1}}{r!} s^r \mathrm{e}^{-\gamma s} \quad \text{for } s \in [0, \infty), \tag{1.5}$$

where γ denotes a positive constant and r denotes a nonnegative integer.

We denote an equilibrium of (1.1) by $\binom{\boldsymbol{x}^*}{\boldsymbol{y}^*}$ where $\boldsymbol{x}^* = (x_1^*, x_2^*, \ldots, x_m^*)^{\mathrm{T}}$, $\boldsymbol{y}^* = (y_1^*, y_2^*, \ldots, y_m^*)^{\mathrm{T}}$, T denotes the transpose of a matrix, and the components x_i^*, y_i^*, $i \in \mathscr{I}$ are governed algebraically by

$$a_i x_i^* = \sum_{j=1}^m b_{ij} S(y_j^*) + I_i, \qquad c_i y_i^* = \sum_{j=1}^m d_{ij} S(x_j^*) + J_i, \quad i \in \mathscr{I}. \tag{1.6}$$

For the convenience of the reader, we provide below the global asymptotic stability of $\binom{\boldsymbol{x}^*}{\boldsymbol{y}^*}$ of the network (1.1) established by Gopalsamy and He [4].

Theorem 1.1. *Let the assumptions* (1.3) *and* (1.4) *hold. Suppose further that*

$$a_i > \sum_{j=1}^m |d_{ji}|, \qquad c_i > \sum_{j=1}^m |b_{ji}|, \quad i \in \mathscr{I}. \tag{1.7}$$

Then the equilibrium $\binom{\boldsymbol{x}^*}{\boldsymbol{y}^*}$ *of the network* (1.1) *is unique and globally asymptotically stable in the sense that*

$$\lim_{t\to\infty} x_i(t) = x_i^*, \quad \lim_{t\to\infty} y_i(t) = y_i^*, \quad i \in \mathscr{I} \tag{1.8}$$

where $x_i(t) = x_i(t, \varphi_i)$, $y_i(t) = y_i(t, \psi_i)$ *for* $i \in \mathscr{I}$, $t > 0$ *denote solutions of the network* (1.1) *corresponding to arbitrary initial values* (1.2).

We note from [4] that the parameters a_i, c_i were chosen as $a_i = c_i = 1$ for all $i \in \mathscr{I}$. Nonetheless, one can still obtain the sufficient condition (1.7) by employing similar Lyapunov functional constructed by Gopalsamy and He [4]. We note further that a bidirectional neural network of the form (1.1) with dynamical characteristics described by Theorem 1.1 can be employed for associative memories, signal or pattern processing, solving optimization problems among others (for details, one can refer [4,21]).

For computer simulation, experimental or computational purposes, it is common to discretize the continuous-time network (1.1). The resulting discrete-time dynamical system is referred to as a discrete-time analogue of (1.1) and it is important that the discrete-time analogue inherits the dynamical characteristics of the continuous-time network (1.1) under mild or no restriction on the discretization step-size. Once this is established, the discrete-time analogue can be used for the above purposes without loss of functional similarity to the continuous-time network (1.1) and preserving any physical or biological reality inherited by the continuous-time network (1.1). With this intention, we formulate in this article a discrete-time analogue of the continuous-time network (1.1). The discrete-time analogue is considered to be a numerical discretization of (1.1) and we study the dynamical characteristics of the discrete-time analogue. We establish that the discrete-time analogue preserves the dynamical characteristics of (1.1) without any restriction on the discretization step-size.

2. Discrete-time formulation

There exists a variety of methods in the literature of numerical analysis and difference equations by which discrete-time analogues of continuous-time dynamical systems can be obtained with an emphasis on the preservation of dynamics. Stuart and Humphries [20] studied discrete-time analogues formulated from conventional methods (such as Euler, Runge–Kutta, multistep, etc.) and identified some analogues which can replicate the dynamical characteristics of their continuous-time counterparts under mild or no restriction on the discretization step-size. Agarwal [1], Mickens [12] and Potts [16–19] formulated discrete-time analogues of certain continuous-time dynamical systems by approximating the continuous-time derivatives with differences involving nonlinear rational functions of the discretization step-size. Without restricting the discretization step-size, they established that some of the discrete-time analogues possess solutions which lie exactly on the solution curves of the corresponding continuous-time systems while others provide the best approximations to their continuous-time counterparts in the sense that the discrete-time analogues inherited similar dynamical characteristics of the corresponding continuous-time systems.

In this section we propose a semi-discretization technique in formulating a discrete-time analogue of the continuous-time network (1.1). This method has been employed elsewhere (see, for instance [5,10,11,14,15]) in the formulation of discrete-time analogues of continuous-time dynamical systems modelling population dynamics and neural networks. For convenience in our study, we adopt the following notations: Let $\mathbb{Z}$ denote the set of all integers; $\mathbb{Z}_0^+ = \{0, 1, 2, \ldots\}$; $[a, b]_{\mathbb{Z}} = \{a, a+1, \ldots, b-1, b\}$ where $a, b \in \mathbb{Z}$, $a \leqslant b$; and $[a, \infty)_{\mathbb{Z}} = \{a, a+1, a+2, \ldots\}$ where $a \in \mathbb{Z}$.

We begin approximating the continuous-time network (1.1) by replacing the integral terms with discrete sums of the form

$$\begin{aligned}
&\int_0^\infty K_{ij}(s)y_j(t-s)\,\mathrm{d}s \approx \sum_{\left[\frac{s}{h}\right]=1}^{\infty} \omega_{ij}(h)K_{ij}\left(\left[\frac{s}{h}\right]h\right) y_j\left(\left[\frac{t}{h}\right]h - \left[\frac{s}{h}\right]h\right),\\
&\int_0^\infty H_{ij}(s)x_j(t-s)\,\mathrm{d}s \approx \sum_{\left[\frac{s}{h}\right]=1}^{\infty} \tilde{\omega}_{ij}(h)H_{ij}\left(\left[\frac{s}{h}\right]h\right) x_j\left(\left[\frac{t}{h}\right]h - \left[\frac{s}{h}\right]h\right)
\end{aligned} \tag{2.1}$$

for $t \in [nh,(n+1)h)$, $s \in [ph,(p+1)h)$, $n \in \mathbb{Z}_0^+$, $p \in \mathbb{Z}^+$, where $[r]$ denotes the integer part of a real number r, $h > 0$ is a fixed number denoting a uniform discretization step-size, $\omega_{ij}(h) > 0$, $\tilde{\omega}_{ij}(h) > 0$ for $h > 0$ and $\omega_{ij}(h) \approx h + O(h^2)$, $\tilde{\omega}_{ij}(h) \approx h + O(h^2)$ for small $h > 0$. We note that $\omega_{ij}(h)$, $\tilde{\omega}_{ij}(h)$ are chosen so that the analogue kernels

$$\mathscr{K}_{ij}\left(\left[\frac{s}{h}\right]h\right) = \omega_{ij}(h)K_{ij}\left(\left[\frac{s}{h}\right]h\right), \quad \mathscr{H}_{ij}\left(\left[\frac{s}{h}\right]h\right) = \tilde{\omega}_{ij}(h)H_{ij}\left(\left[\frac{s}{h}\right]h\right) \tag{2.2}$$

satisfy certain properties given in (2.8) below. With (2.1) and (2.2), we approximate (1.1) by differential equations with piecewise constant arguments of the form

$$\begin{aligned}
\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} S\left(\sum_{\left[\frac{s}{h}\right]=1}^{\infty} \mathscr{K}_{ij}\left(\left[\frac{s}{h}\right]h\right) y_j\left(\left[\frac{t}{h}\right]h - \left[\frac{s}{h}\right]h\right)\right) + I_i,\\
\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} &= -c_i y_i(t) + \sum_{j=1}^{m} d_{ij} S\left(\sum_{\left[\frac{s}{h}\right]=1}^{\infty} \mathscr{H}_{ij}\left(\left[\frac{s}{h}\right]h\right) x_j\left(\left[\frac{t}{h}\right]h - \left[\frac{s}{h}\right]h\right)\right) + J_i
\end{aligned} \tag{2.3}$$

for $i \in \mathscr{I}$, $t \in [nh,(n+1)h)$, $s \in [ph,(p+1)h)$, $n \in \mathbb{Z}_0^+$, $p \in \mathbb{Z}^+$. Noting that $[\frac{t}{h}] = n$, $[\frac{s}{h}] = p$ and adopting the notation $u(n) = u(nh)$, we rewrite (2.3) as

$$\begin{aligned}
\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} S\left(\sum_{p=1}^{\infty} \mathscr{K}_{ij}(p) y_j(n-p)\right) + I_i,\\
\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} &= -c_i y_i(t) + \sum_{j=1}^{m} d_{ij} S\left(\sum_{p=1}^{\infty} \mathscr{H}_{ij}(p) x_j(n-p)\right) + J_i
\end{aligned} \tag{2.4}$$

for $i \in \mathscr{I}$, $t \in [nh,(n+1)h)$, $n \in \mathbb{Z}_0^+$. We integrate system (2.4) over the interval $[nh,t)$ where $t<(n+1)h$ to get

$$\begin{aligned} x_i(t)\mathrm{e}^{a_i t} &= x_i(n)\mathrm{e}^{a_i nh} + \left(\frac{\mathrm{e}^{a_i t}-\mathrm{e}^{a_i nh}}{a_i}\right)\left[\sum_{j=1}^{m} b_{ij} S\left(\sum_{p=1}^{\infty}\mathscr{K}_{ij}(p)y_j(n-p)\right)+I_i\right], \\ y_i(t)\mathrm{e}^{c_i t} &= y_i(n)\mathrm{e}^{c_i nh} + \left(\frac{\mathrm{e}^{c_i t}-\mathrm{e}^{c_i nh}}{c_i}\right)\left[\sum_{j=1}^{m} d_{ij} S\left(\sum_{p=1}^{\infty}\mathscr{H}_{ij}(p)x_j(n-p)\right)+J_i\right] \end{aligned} \tag{2.5}$$

for $i \in \mathscr{I}$, $t \in [nh,(n+1)h)$, $n \in \mathbb{Z}_0^+$. Allowing $t \to (n+1)h$ in (2.5) and simplifying, we obtain a discrete-time analogue of the continuous-time network (1.1) given by

$$\begin{aligned} x_i(n+1) &= \mathrm{e}^{-a_i h}x_i(n) + \theta_i(h)\sum_{j=1}^{m} b_{ij} S\left(\sum_{p=1}^{\infty}\mathscr{K}_{ij}(p)y_j(n-p)\right)+\theta_i(h)I_i, \\ y_i(n+1) &= \mathrm{e}^{-c_i h}y_i(n) + \phi_i(h)\sum_{j=1}^{m} d_{ij} S\left(\sum_{p=1}^{\infty}\mathscr{H}_{ij}(p)x_j(n-p)\right)+\phi_i(h)J_i \end{aligned} \tag{2.6}$$

for $i \in \mathscr{I}$, $n \in \mathbb{Z}_0^+$ where

$$\theta_i(h) = \frac{1-\mathrm{e}^{-a_i h}}{a_i}, \qquad \phi_i(h) = \frac{1-\mathrm{e}^{-c_i h}}{c_i}.$$

One can verify that $\theta_i(h)>0$, $\phi_i(h)>0$ if $a_i, c_i, h>0$ and $\theta_i(h) \approx h+O(h^2)$, $\phi_i(h) \approx h+O(h^2)$ for small $h>0$. One can show that the discrete-time analogue (2.6) converges to the continuous-time network (1.1) when $h \to 0^+$.

The initial conditions associated with the analogue (2.6) are given in the form

$$x_i(l) = \varphi_i(l), \qquad y_i(l) = \psi_i(l) \quad \text{for } i \in \mathscr{I}, \quad l \in (-\infty,0]_{\mathbb{Z}}, \tag{2.7}$$

where $\varphi_i(l)$, $\psi_i(l)$ defined for $l \in (-\infty,0]_{\mathbb{Z}}$ denote sequences of real numbers and

$$\sup_{l\in(-\infty,0]_{\mathbb{Z}}} |\varphi_i(l)| < \infty, \qquad \sup_{l\in(-\infty,0]_{\mathbb{Z}}} |\psi_i(l)| < \infty, \quad i \in \mathscr{I}.$$

Notice that (2.7) is a consequence of discretizing the initial values (1.2) supplemented to the continuous-time network (1.1).

The kernels $\mathscr{K}_{ij}(\cdot)$, $\mathscr{H}_{ij}(\cdot)$ in the discrete-time analogue (2.6) are assumed to satisfy the following properties:

(i) $\mathscr{K}_{ij}: \mathbb{Z}^+ \to [0,\infty)$, $\quad \mathscr{H}_{ij}: \mathbb{Z}^+ \to [0,\infty)$,

(ii) $\displaystyle\sum_{p=1}^{\infty}\mathscr{K}_{ij}(p) = 1$, $\quad \displaystyle\sum_{p=1}^{\infty}\mathscr{H}_{ij}(p) = 1$,

(iii) there exists a real number $v > 1$ such that

$$\sum_{p=1}^{\infty}\mathscr{K}_{ij}(p)v^p < \infty, \quad \sum_{p=1}^{\infty}\mathscr{H}_{ij}(p)v^p < \infty. \tag{2.8}$$

We remark that the properties in (2.8) are analogous to those given by (1.4). We consider an example, say, $K(s) = \gamma e^{-\gamma s}$ for $s \in [0, \infty)$ depicted from (1.5). This kernel satisfies the properties in (1.4). An analogue of $K(\cdot)$ is given by $\mathscr{K}(p) = \omega(h)\gamma e^{-\gamma ph}$ for $p \in \mathbb{Z}^+$, where $\omega(h) = \frac{1-e^{-\gamma h}}{\gamma e^{-\gamma h}}$. We recall that the step-size h is a fixed positive number. One can verify that the analogue kernel $\mathscr{K}(\cdot)$ satisfies the properties (i) and (ii) of (2.8). Moreover, by letting $v = e^{\mu h}$, where μ is a constant satisfying $0 < \mu < \gamma$, one observes that

$$\sum_{p=1}^{\infty}\mathscr{K}(p)v^p = \sum_{p=1}^{\infty}\frac{1-e^{-\gamma h}}{\gamma e^{-\gamma h}}\gamma e^{-\gamma ph}e^{\mu ph} = \frac{1-e^{-\gamma h}}{e^{-\gamma h}}\sum_{p=1}^{\infty}e^{-(\gamma-\mu)ph} < \infty$$

and this justifies the analogue kernel $\mathscr{K}(\cdot)$ satisfying the property (iii) of (2.8).

An equilibrium of the analogue (2.6) satisfies

$$x_i^* = e^{-a_i h}x_i^* + \theta_i(h)\sum_{j=1}^{m}b_{ij}S\left(\sum_{p=1}^{\infty}\mathscr{K}_{ij}(p)y_j^*\right) + \theta_i(h)I_i,$$

$$y_i^* = e^{-c_i h}y_i^* + \phi_i(h)\sum_{j=1}^{m}d_{ij}S\left(\sum_{p=1}^{\infty}\mathscr{H}_{ij}(p)x_j^*\right) + \phi_i(h)J_i$$

and upon using (2.8) in the above one obtains the system (1.6). This implies that for any choice of the positive step-size h the analogue (2.6) preserves the equilibrium $\binom{x^*}{y^*}$ of the continuous-time network (1.1).

3. Exponential stability

We proceed in this section to establish the exponential stability of the equilibrium $\binom{x^*}{y^*}$ of the discrete-time analogue (2.6).

Theorem 3.1. *Let the positive step-size h be fixed and let the assumptions* (1.3) *and* (2.8) *be satisfied. Suppose the condition* (1.7) *holds. Then the equilibrium $\binom{x^*}{y^*}$ of the analogue* (2.6) *exists and unique. Moreover, there exist real constants $\beta \geqslant 1$ and $1 < \lambda < v$, where the constant $v > 1$ is defined in* (2.8), *such that*

$$\sum_{i=1}^{m}\{|x_i(n) - x_i^*| + |y_i(n) - y_i^*|\}$$
$$\leqslant \beta\left(\frac{1}{\lambda}\right)^n\sum_{i=1}^{m}\left\{\left(\sup_{l\in(-\infty,0]_{\mathbb{Z}}}|x_i(l) - x_i^*|\right) + \left(\sup_{l\in(-\infty,0]_{\mathbb{Z}}}|y_i(l) - y_i^*|\right)\right\} \tag{3.1}$$

for $n \in \mathbb{Z}^+$ *where* $x_i(n) = x_i(n, \varphi_i)$, $y_i(n) = y_i(n, \psi_i)$ *for* $i \in \mathscr{I}$, $n \in \mathbb{Z}^+$ *denote solutions of the analogue* (2.6) *corresponding to an arbitrary initial condition* (2.7).

Proof. We note that the signal response function $S(u) = \tanh(u)$, $u \in \mathbb{R}$ satisfies the following properties: $|S(u)| \leqslant 1$ for all $u \in \mathbb{R}$ and $|S(u) - S(v)| \leqslant |u - v|$ for all $u, v \in \mathbb{R}$.

We show the existence and uniqueness of the equilibrium $\binom{\boldsymbol{x}^*}{\boldsymbol{y}^*}$ of the analogue (2.6). We consider a mapping

$$\boldsymbol{P}(\theta) = \begin{pmatrix} \boldsymbol{F}(\theta) \\ \boldsymbol{G}(\theta) \end{pmatrix} \quad \text{for } \theta = x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m \in \mathbb{R}^{2m}, \tag{3.2}$$

where $\boldsymbol{F}(\theta) = (F_1(\theta), F_2(\theta), \ldots, F_m(\theta))^{\mathrm{T}}$ and $\boldsymbol{G}(\theta) = (G_1(\theta), G_2(\theta), \ldots, G_m(\theta))^{\mathrm{T}}$ with

$$x_i = F_i(\theta) = \frac{1}{a_i}\left(\sum_{j=1}^{m} b_{ij} S(y_j) + I_i\right), \qquad y_i = G_i(\theta) = \frac{1}{c_i}\left(\sum_{j=1}^{m} d_{ij} S(x_j) + J_i\right) \tag{3.3}$$

for $i \in \mathscr{I}$. By applying the assumption (1.3) and the boundedness of $S(\cdot)$ we obtain from (3.3) that

$$|F_i(\theta)| \leqslant \frac{1}{a_i}\left(\sum_{j=1}^{m} |b_{ij}| + |I_i|\right) \leqslant \sigma, \qquad |G_i(\theta)| \leqslant \frac{1}{c_i}\left(\sum_{j=1}^{m} |d_{ij}| + |I_i|\right) \leqslant \sigma \tag{3.4}$$

for all $i \in \mathscr{I}$, where $\sigma = \max\{\sigma_1, \sigma_2\}$ and

$$\sigma_1 = \max_{i \in \mathscr{I}} \left\{ \frac{1}{a_i}\left(\sum_{j=1}^{m} |b_{ij}| + |I_i|\right) \right\}, \qquad \sigma_2 = \max_{i \in \mathscr{I}} \left\{ \frac{1}{c_i}\left(\sum_{j=1}^{m} |d_{ij}| + |J_i|\right) \right\}.$$

We gather from (3.2)–(3.4) that

$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in [-\sigma, \sigma]^{2m} \Rightarrow \boldsymbol{P}(\theta) = \begin{pmatrix} \boldsymbol{F}(\theta) \\ \boldsymbol{G}(\theta) \end{pmatrix} \in [-\sigma, \sigma]^{2m},$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)^{\mathrm{T}}$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_m)^{\mathrm{T}}$. Since $S(u)$ is continuous for $u \in \mathbb{R}$ it follows from (3.3) that $F_i(\theta)$, $G_i(\theta)$ are continuous for $\theta \in \mathbb{R}^{2m}$. This implies that the mapping $\boldsymbol{P} : [-\sigma, \sigma]^{2m} \to [-\sigma, \sigma]^{2m}$ is continuous. From the Brouwer's fixed point theorem, we assert that there exists at least one $\binom{\boldsymbol{x}^*}{\boldsymbol{y}^*} \in [-\sigma, \sigma]^{2m}$ which is a fixed point of the mapping $\boldsymbol{P}$. The fixed point $\binom{\boldsymbol{x}^*}{\boldsymbol{y}^*}$ is the equilibrium of the analogue (2.6). In the following, we show that the condition (1.7) guarantees the uniqueness of $\binom{\boldsymbol{x}^*}{\boldsymbol{y}^*}$. Suppose there exists another equilibrium $\binom{\boldsymbol{u}^*}{\boldsymbol{v}^*}$ of the analogue (2.6). From the algebraic system (1.6) we have

$$a_i(x_i^* - u_i^*) = \sum_{j=1}^{m} b_{ij}(S(y_j^*) - S(v_j^*)), \qquad c_i(y_i^* - v_i^*) = \sum_{j=1}^{m} d_{ij}(S(x_j^*) - S(u_j^*))$$

for $i \in \mathscr{I}$ and by using the assumption (1.3) and the Lipschitz property of $S(\cdot)$ in the above we obtain the following inequalities given by

$$a_i|x_i^* - u_i^*| \leqslant \sum_{j=1}^{m} |b_{ij}||y_j^* - v_j^*|, \qquad c_i^*|y_i^* - v_i^*| \leqslant \sum_{j=1}^{m} |d_{ij}||x_j^* - u_j^*|, \quad i \in \mathscr{I}. \tag{3.5}$$

It follows from (3.5) that

$$\sum_{i=1}^{m}\left\{\left(a_i-\sum_{j=1}^{m}|d_{ji}|\right)|x_i^*-u_i^*|+\left(c_i-\sum_{j=1}^{m}|b_{ji}|\right)|y_i^*-v_i^*|\right\}\leqslant 0.$$

By using the condition (1.7) we deduce from the above that $x_i^*=u_i^*$ and $y_i^*=v_i^*$ for all $i\in\mathscr{I}$. This establishes the uniqueness of the equilibrium $\binom{x^*}{y^*}$ of the analogue (2.6).

Next, we prove the global exponential stability of the equilibrium $\binom{x^*}{y^*}$ of the analogue (2.6). By applying the assumptions (1.3), (2.8) and the Lipschitz property of $S(\cdot)$, we obtain from (2.6) that

$$\begin{aligned}|x_i(n+1)-x_i^*| &\leqslant \mathrm{e}^{-a_ih}|x_i(n)-x_i^*|+\theta_i(h)\sum_{j=1}^{m}|b_{ij}|\sum_{p=1}^{\infty}\mathscr{K}_{ij}(p)|y_j(n-p)-y_j^*|,\\ |y_i(n+1)-y_i^*| &\leqslant \mathrm{e}^{-c_ih}|y_i(n)-y_i^*|+\phi_i(h)\sum_{j=1}^{m}|d_{ij}|\sum_{p=1}^{\infty}\mathscr{H}_{ij}(p)|x_j(n-p)-x_j^*|\end{aligned}\tag{3.6}$$

for $i\in\mathscr{I}$, $n\in\mathbb{Z}_0^+$. Let us consider functions $\mathscr{F}_i(\cdot)$, $\mathscr{G}_i(\cdot)$, $i\in\mathscr{I}$ defined by

$$\begin{aligned}\mathscr{F}_i(\zeta_i) &= 1-\zeta_i\mathrm{e}^{-a_ih}-\theta_i(h)\sum_{j=1}^{m}|d_{ji}|\sum_{p=1}^{\infty}\mathscr{H}_{ji}(p)\zeta_i^{p+1},\quad \zeta_i\in[1,v],\\ \mathscr{G}_i(\xi_i) &= 1-\xi_i\mathrm{e}^{-c_ih}-\phi_i(h)\sum_{j=1}^{m}|b_{ji}|\sum_{p=1}^{\infty}\mathscr{K}_{ji}(p)\xi_i^{p+1},\quad \xi_i\in[1,v]\end{aligned}\tag{3.7}$$

where the constant $v>1$ is defined in (2.8). We note from the condition (1.7) that

$$a_i-\sum_{j=1}^{m}|d_{ji}|\geqslant\rho,\qquad c_i-\sum_{j=1}^{m}|b_{ji}|\geqslant\rho\quad\text{for all } i\in\mathscr{I}\tag{3.8}$$

where $\rho=\min\{\rho_1,\rho_2\}$ and

$$\rho_1=\min_{i\in\mathscr{I}}\left\{a_i-\sum_{j=1}^{m}|d_{ji}|\right\}>0,\qquad \rho_2=\min_{i\in\mathscr{I}}\left\{c_i-\sum_{j=1}^{m}|b_{ji}|\right\}>0.$$

We observe from (3.7) that

$$\begin{aligned}\mathscr{F}_i(1) &= 1-\mathrm{e}^{-a_ih}-\theta_i(h)\sum_{j=1}^{m}|d_{ji}|\\ &= \theta_i(h)\left(a_i-\sum_{j=1}^{m}|d_{ji}|\right)\geqslant\min_{i\in\mathscr{I}}\{\theta_i(h)\}\rho>0,\end{aligned}$$

$$\mathscr{G}_i(1) = 1 - \mathrm{e}^{-c_i h} - \phi_i(h)\sum_{j=1}^{m}|b_{ji}|$$
$$= \phi_i(h)\left(c_i - \sum_{j=1}^{m}|b_{ji}|\right) \geqslant \min_{i\in\mathscr{I}}\{\phi_i(h)\}\rho > 0.$$

We observe further that $\mathscr{F}_i(\zeta_i)$, $\mathscr{G}_i(\xi_i)$ are continuous for $\zeta_i, \xi_i \in [1, v]$ and $\mathscr{F}_i(\zeta_i) \to -\infty$ as $\zeta_i \to v^-$, $\mathscr{G}_i(\xi_i) \to -\infty$ as $\xi_i \to v^-$. It follows that there exist $\hat{\zeta}_i, \hat{\xi}_i \in (1, v)$ such that $\mathscr{F}_i(\hat{\zeta}_i) = 0$, $\mathscr{G}_i(\hat{\xi}_i) = 0$ for $i \in \mathscr{I}$. We choose

$$\lambda = \min\{\hat{\zeta}_1, \hat{\zeta}_2, \ldots, \hat{\zeta}_m, \hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_m\},$$

where obviously $\lambda \in (1, v)$, and as a result we have $\mathscr{F}_i(\lambda) \geqslant 0$, $\mathscr{G}_i(\lambda) \geqslant 0$ for all $i \in \mathscr{I}$ implying that

$$\left.\begin{aligned} &\lambda\mathrm{e}^{-a_i h} + \theta_i(h)\sum_{j=1}^{m}|d_{ji}|\sum_{p=1}^{\infty}\mathscr{H}_{ji}(p)\lambda^{p+1} \leqslant 1,\\ &\lambda\mathrm{e}^{-c_i h} + \phi_i(h)\sum_{j=1}^{m}|b_{ji}|\sum_{p=1}^{\infty}\mathscr{K}_{ji}(p)\lambda^{p+1} \leqslant 1 \end{aligned}\right\} \quad \text{for all } i \in \mathscr{I}. \tag{3.9}$$

Let us consider nonnegative sequences $X_i(n)$, $Y_i(n)$ for $i \in \mathscr{I}$, $n \in \mathbb{Z}$ defined by

$$X_i(n) = \lambda^n \frac{|x_i(n) - x_i^*|}{\theta_i(h)}, \qquad Y_i(n) = \lambda^n \frac{|y_i(n) - y_i^*|}{\phi_i(h)} \tag{3.10}$$

and from (3.6) we derive the following system:

$$X_i(n+1) \leqslant \lambda\mathrm{e}^{-a_i h}X_i(n) + \sum_{j=1}^{m}|b_{ij}|\phi_j(h)\sum_{p=1}^{\infty}\mathscr{K}_{ij}(p)\lambda^{p+1}Y_j(n-p),$$
$$Y_i(n+1) \leqslant \lambda\mathrm{e}^{-c_i h}Y_i(n) + \sum_{j=1}^{m}|d_{ij}|\theta_j(h)\sum_{p=1}^{\infty}\mathscr{H}_{ij}(p)\lambda^{p+1}X_j(n-p) \tag{3.11}$$

for $i \in \mathscr{I}$, $n \in \mathbb{Z}_0^+$.

We construct a Lyapunov-type sequence given by

$$V(n) = \sum_{i=1}^{m}\left\{X_i(n) + Y_i(n) + \sum_{j=1}^{m}|b_{ij}|\phi_j(h)\sum_{p=1}^{\infty}\mathscr{K}_{ij}(p)\lambda^{p+1}\sum_{q=n-p}^{n-1}Y_j(q)\right.$$
$$\left. + \sum_{j=1}^{m}|d_{ij}|\theta_j(h)\sum_{p=1}^{\infty}\mathscr{H}_{ij}(p)\lambda^{p+1}\sum_{q=n-p}^{n-1}X_j(q)\right\}, \quad n \in \mathbb{Z}_0^+. \tag{3.12}$$

It can be verified that $V(n) > 0$ for all $n \in \mathbb{Z}_0^+$. We note that if $V(0)$ is finite (i.e., bounded) and $V(n)$ is nonincreasing for all $n \in \mathbb{Z}^+$ then $V(n)$ will remain bounded and defined for all $n \in \mathbb{Z}^+$. We derive from (3.12) that

$$V(0) \leqslant \sum_{i=1}^{m} \left\{ \left(1 + \theta_i(h) \sum_{j=1}^{m} |d_{ji}| \sum_{p=1}^{\infty} \mathscr{H}_{ji}(p) \lambda^{p+1} p \right) \left(\sup_{q \in (-\infty,0]_{\mathbb{Z}}} X_i(q) \right) \right\}$$
$$+ \sum_{i=1}^{m} \left\{ \left(1 + \phi_i(h) \sum_{j=1}^{m} |b_{ji}| \sum_{p=1}^{\infty} \mathscr{K}_{ji}(p) \lambda^{p+1} p \right) \left(\sup_{q \in (-\infty,0]_{\mathbb{Z}}} Y_i(q) \right) \right\}. \tag{3.13}$$

It follows from (2.7) and (3.10) that

$$\sup_{q \in (-\infty,0]_{\mathbb{Z}}} X_i(q) < \infty, \qquad \sup_{q \in (-\infty,0]_{\mathbb{Z}}} Y_i(q) < \infty \quad \text{for } i \in \mathscr{I}.$$

We recall the assumption (2.8) and we note that the constant λ satisfies $1 < \lambda < v$. Let $p \in \mathbb{Z}^+$, $v = \mathrm{e}^{\mu}$, where $\mu > 0$ and let $\lambda = \mathrm{e}^{\delta}$ where clearly $0 < \delta < \mu$. We observe that

$$p\lambda^p = p\mathrm{e}^{\delta p} \leqslant \mathrm{e}^{\mu p/2} \mathrm{e}^{\mu p/2} \quad \text{for all large } p \in \mathbb{Z}^+.$$

Consequently,

$$\sum_{p=1}^{\infty} p \mathscr{K}_{ji}(p) \lambda^p = \sum_{p=1}^{\infty} p \mathscr{K}_{ji}(p) \mathrm{e}^{\delta p} \leqslant \sum_{p=1}^{\infty} \mathscr{K}_{ji}(p) \mathrm{e}^{\mu p} < \infty,$$
$$\sum_{p=1}^{\infty} p \mathscr{H}_{ji}(p) \lambda^p = \sum_{p=1}^{\infty} p \mathscr{H}_{ji}(p) \mathrm{e}^{\delta p} \leqslant \sum_{p=1}^{\infty} \mathscr{H}_{ji}(p) \mathrm{e}^{\mu p} < \infty$$

due to the property (iii) of (2.8). With these observations we assert from (3.13) that $V(0)$ is bounded.

Next, we calculate the difference $\triangle V(n) = V(n+1) - V(n)$ for $n \in \mathbb{Z}_0^+$ along the solutions of (3.11) and we obtain

$$\triangle V(n) \leqslant -\sum_{i=1}^{m} \left(1 - \lambda \mathrm{e}^{-a_i h} - \theta_i(h) \sum_{j=1}^{m} |d_{ji}| \sum_{p=1}^{\infty} \mathscr{H}_{ji}(p) \lambda^{p+1} \right) X_i(n)$$
$$- \sum_{i=1}^{m} \left(1 - \lambda \mathrm{e}^{-c_i h} - \phi_i(h) \sum_{j=1}^{m} |b_{ji}| \sum_{p=1}^{\infty} \mathscr{K}_{ji}(p) \lambda^{p+1} \right) Y_i(n), \quad n \in \mathbb{Z}_0^+$$

and on behalf of (3.9) we deduce that $\triangle V(n) \leqslant 0$ for all $n \in \mathbb{Z}_0^+$. This in turn implies that $V(n)$ is nonincreasing for all $n \in \mathbb{Z}^+$ or equivalently $V(n) \leqslant V(0) < \infty$ for all $n \in \mathbb{Z}^+$. We gather from (3.12) and (3.13) that

$$\sum_{i=1}^{m} \{X_i(n) + Y_i(n)\} \leqslant \beta_1 \sum_{i=1}^{m} \left(\sup_{q \in (-\infty,0]_{\mathbb{Z}}} X_i(q) \right) + \beta_2 \sum_{i=1}^{m} \left(\sup_{q \in (-\infty,0]_{\mathbb{Z}}} Y_i(q) \right) \tag{3.14}$$

for all $n \in \mathbb{Z}^+$ where

$$\beta_1 = \max_{i \in \mathscr{I}} \left\{ 1 + \theta_i(h) \sum_{j=1}^{m} |d_{ji}| \sum_{p=1}^{\infty} \mathscr{K}_{ji}(p) \lambda^{p+1} p \right\} \geqslant 1,$$

$$\beta_2 = \max_{i \in \mathscr{I}} \left\{ 1 + \phi_i(h) \sum_{j=1}^{m} |b_{ji}| \sum_{p=1}^{\infty} \mathscr{K}_{ji}(p) \lambda^{p+1} p \right\} \geqslant 1.$$

Recalling definition (3.10) we have from (3.14) that

$$\sum_{i=1}^{m} \{ |x_i(n) - x_i^*| + |y_i(n) - y_i^*| \}$$

$$\leqslant \beta \left(\frac{1}{\lambda}\right)^n \sum_{i=1}^{m} \left\{ \left(\sup_{l \in (-\infty,0]_{\mathbb{Z}}} |x_i(l) - x_i^*| \right) + \left(\sup_{l \in (-\infty,0]_{\mathbb{Z}}} |y_i(l) - y_i^*| \right) \right\}$$

for all $n \in \mathbb{Z}^+$ where

$$\beta = \frac{\max\{\max_{i \in \mathscr{I}}\{\theta_i(h)\}, \max_{i \in \mathscr{I}}\{\phi_i(h)\}\}}{\min\{\min_{i \in \mathscr{I}}\{\theta_i(h)\}, \min_{i \in \mathscr{I}}\{\phi_i(h)\}\}} \max\{\beta_1, \beta_2\} \geqslant 1.$$

This establishes the global exponential stability of the equilibrium $\binom{x^*}{y^*}$ of the analogue (2.6) and hence the proof is complete. $\square$

In the following result, we obtain another set of easily verifiable sufficient conditions for the global exponential stability of the equilibrium $\binom{x^*}{y^*}$ of the analogue (2.6). We remark that these conditions have not been obtained in the literature of continuous-time bidirectional neural networks.

Theorem 3.2. *Assume that the discretization step-size is fixed and positive and let the assumptions* (1.3) *and* (2.8) *be satisfied. Suppose*

$$a_i > \sum_{j=1}^{m} |b_{ij}|, \qquad c_i > \sum_{j=1}^{m} |d_{ij}|, \quad i \in \mathscr{I}. \tag{3.15}$$

Then the equilibrium $\binom{x^*}{y^*}$ *of the analogue* (2.6) *exists and unique. Moreover, there exists a real constant* α *satisfying* $1 < \alpha < v$, *where the constant* $v > 1$ *is defined in* (2.8), *such that*

$$|x_i(n) - x_i^*| \leqslant M \left(\frac{1}{\alpha}\right)^n, \qquad |y_i(n) - y_i^*| \leqslant M \left(\frac{1}{\alpha}\right)^n \tag{3.16}$$

for all $i \in \mathscr{I}$, $n \in \mathbb{Z}^+$ *where* $M = \max\{M_1, M_2\}$ *and*

$$0 < M_1 = \max_{i \in \mathscr{I}} \left\{ \sup_{l \in (-\infty,0]_{\mathbb{Z}}} |x_i(l) - x_i^*| \right\} < \infty,$$

$$0 < M_2 = \max_{i \in \mathscr{I}} \left\{ \sup_{l \in (-\infty,0]_{\mathbb{Z}}} |y_i(l) - y_i^*| \right\} < \infty.$$

Proof. The existence of an equilibrium $\binom{x^*}{y^*}$ of the analogue (2.6) follows from Theorem 3.1. Suppose there exists another equilibrium $\binom{u^*}{v^*}$ of (2.6). We shall see that $x_i^* = u_i^*$ and $y_i^* = v_i^*$ componentwise for $i \in \mathscr{I}$. Let us assume for the sake of contradiction that there exist kth-components of $\boldsymbol{x}^*$ and $\boldsymbol{u}^*$ such that $x_k^* \neq u_k^*$. We have from the system (3.5) that

$$a_k|x_k^* - u_k^*| \leqslant 0 \quad \text{and} \quad 0 \leqslant |d_{ik}||x_k^* - u_k^*|, \quad i \in \mathscr{I}.$$

Since $a_k > 0$ we must have $x_k^* = u_k^*$ and this violates the assumption that $x_k^* \neq u_k^*$. With this we assert that the equilibrium $\binom{x^*}{y^*}$ is unique.

Let us consider functions $\mathscr{P}_i(\cdot)$, $\mathscr{Q}_i(\cdot)$, $i \in \mathscr{I}$ defined by

$$\begin{aligned} \mathscr{P}_i(\zeta_i) &= 1 - \zeta_i e^{-a_i h} - \theta_i(h)\sum_{j=1}^{m}|b_{ij}|\sum_{p=1}^{\infty}\mathscr{K}_{ij}(p)\zeta_i^{p+1}, \quad \zeta_i \in [1, v], \\ \mathscr{Q}_i(\xi_i) &= 1 - \xi_i e^{-c_i h} - \phi_i(h)\sum_{j=1}^{m}|d_{ij}|\sum_{p=1}^{\infty}\mathscr{H}_{ij}(p)\xi_i^{p+1}, \quad \xi_i \in [1, v]. \end{aligned} \tag{3.17}$$

We note from the condition (3.15) that

$$a_i - \sum_{j=1}^{m}|b_{ij}| \geqslant \eta, \qquad c_i - \sum_{j=1}^{m}|d_{ij}| \geqslant \eta \tag{3.18}$$

for all $i \in \mathscr{I}$ where $\eta = \min\{\eta_1, \eta_2\}$ and

$$\eta_1 = \min_{i\in\mathscr{I}}\left\{a_i - \sum_{j=1}^{m}|b_{ij}|\right\} > 0, \qquad \eta_2 = \min_{i\in\mathscr{I}}\left\{c_i - \sum_{j=1}^{m}|d_{ij}|\right\} > 0.$$

By applying (2.8) and (3.18) to (3.17) one has $\mathscr{P}_i(1) \geqslant \min_{i\in\mathscr{I}}\{\theta_i(h)\}\eta > 0$, $\mathscr{Q}_i(1) \geqslant \min_{i\in\mathscr{I}}\{\phi_i(h)\}\eta > 0$ and by similar arguments of Theorem 3.1 one can assert that there exist $\tilde{\zeta}_i, \tilde{\xi}_i \in (1, v)$ which are roots of the functions $\mathscr{P}_i(\cdot)$, $\mathscr{Q}_i(\cdot)$, $i \in \mathscr{I}$, respectively. By choosing

$$\alpha = \min\{\tilde{\zeta}_1, \tilde{\zeta}_2, \ldots, \tilde{\zeta}_m, \tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_m\},$$

where obviously $\alpha \in (1, v)$, one deduces from (3.17) that

$$\begin{aligned} &\alpha e^{-a_i h} + \theta_i(h)\sum_{j=1}^{m}|b_{ij}|\sum_{p=1}^{\infty}\mathscr{K}_{ij}(p)\alpha^{p+1} \leqslant 1, \\ &\alpha e^{-c_i h} + \phi_i(h)\sum_{j=1}^{m}|d_{ij}|\sum_{p=1}^{\infty}\mathscr{H}_{ij}(p)\alpha^{p+1} \leqslant 1 \end{aligned} \tag{3.19}$$

for all $i \in \mathscr{I}$.

Now we define nonnegative sequences $U_i(n)$, $V_i(n)$ for $i \in \mathscr{I}$, $n \in \mathbb{Z}$ as

$$U_i(n) = \alpha^n|x_i(n) - x_i^*|, \qquad V_i(n) = \alpha^n|y_i(n) - y_i^*|. \tag{3.20}$$

We use (3.20) in the system (3.6) and we derive that

$$\begin{aligned} U_i(n+1) &\leqslant \alpha e^{-a_i h} U_i(n) + \theta_i(h) \sum_{j=1}^{m} |b_{ij}| \sum_{p=1}^{\infty} \mathscr{K}_{ij}(p) \alpha^{p+1} \left(\sup_{l \in (-\infty, n-1]_{\mathbb{Z}}} V_j(l) \right), \\ V_i(n+1) &\leqslant \alpha e^{-c_i h} V_i(n) + \phi_i(h) \sum_{j=1}^{m} |d_{ij}| \sum_{p=1}^{\infty} \mathscr{H}_{ij}(p) \alpha^{p+1} \left(\sup_{l \in (-\infty, n-1]_{\mathbb{Z}}} U_j(l) \right) \end{aligned} \tag{3.21}$$

for $i \in \mathscr{I}$, $n \in \mathbb{Z}_0^+$. We remark that system (3.21) denotes a discrete version of a system of Halanay-type inequalities. The interested reader can refer to [3,6] for expositions and various applications of continuous-time Halanay-type inequalities; one can refer to [2,13] for literature related to discrete-time Halanay-type inequalities.

It follows from the definition (3.20) and the definition of the constant $M \in (0, \infty)$ that

$$U_i(l) \leqslant M, \quad V_i(l) \leqslant M \quad \text{for all } i \in \mathscr{I}, \ l \in (-\infty, 0]_{\mathbb{Z}}.$$

We claim that

$$U_i(n) \leqslant M, \quad V_i(n) \leqslant M \quad \text{for all } i \in \mathscr{I}, \ n \in \mathbb{Z}^+. \tag{3.22}$$

Suppose the claim is not valid in the sense that there is a kth-component among $U_i(\cdot)$ and a first time $n_1 \in \mathbb{Z}^+$ such that

$$U_k(n) \leqslant M \quad \text{for } n \in (-\infty, n_1 - 1]_{\mathbb{Z}} \quad \text{and} \quad U_k(n_1) > M,$$

while

$$U_i(n) \leqslant M \quad \text{for } i \in \mathscr{I}, \ i \neq k, \ n \in (-\infty, n_1]_{\mathbb{Z}},$$

$$V_i(n) \leqslant M \quad \text{for } i \in \mathscr{I}, \ n \in (-\infty, n_1]_{\mathbb{Z}}.$$

We obtain from (3.21) that

$$\begin{aligned} M < U_k(n_1) &\leqslant \alpha e^{-a_k h} U_k(n_1 - 1) + \theta_k(h) \sum_{j=1}^{m} |b_{kj}| \sum_{p=1}^{\infty} \mathscr{K}_{kj}(p) \alpha^{p+1} \left(\sup_{l \in (-\infty, n_1-2]_{\mathbb{Z}}} V_j(l) \right) \\ &\leqslant \left(\alpha e^{-a_k h} + \theta_k(h) \sum_{j=1}^{m} |b_{kj}| \sum_{p=1}^{\infty} \mathscr{K}_{kj}(p) \alpha^{p+1} \right) M \end{aligned}$$

and on behalf of (3.19) we are led to a contradiction in the sense that $M < U_k(n_1) \leqslant M$. Thus the claim (3.22) is valid. From (3.20) and (3.22) we deduce that

$$|x_i(n) - x_i^*| \leqslant M \left(\frac{1}{\alpha} \right)^n, \qquad |y_i(n) - y_i^*| \leqslant M \left(\frac{1}{\alpha} \right)^n$$

for all $i \in \mathscr{I}$, $n \in \mathbb{Z}^+$. Since the constant α satisfies $1 < \alpha < v$ and the initial values are arbitrary, we conclude from the above that the equilibrium $\binom{x^*}{y^*}$ is globally exponentially stable. The proof is complete. □

4. Computer simulations

In this section, we provide several computer simulations of a bidirectional neural network with delays distributed over finite intervals; the continuous-time network is governed by

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_i x_i(t) + \sum_{j=1}^{2} b_{ij} \tanh\left(\int_0^{\tau_{ij}} K_{ij}(s) y_j(t-s)\,\mathrm{d}s\right) + I_i,\\ \frac{\mathrm{d}y_i(t)}{\mathrm{d}t} &= -c_i y_i(t) + \sum_{j=1}^{2} d_{ij} \tanh\left(\int_0^{\sigma_{ij}} H_{ij}(s) x_j(t-s)\,\mathrm{d}s\right) + J_i\end{aligned} \tag{4.1}$$

for $i = 1,2,\ t > 0$, where the synaptic weights $b_{ij},\ d_{ij}$, the exogeneous inputs $I_i,\ J_i$ and the delay parameters $\tau_{ij},\ \sigma_{ij}$ are assumed to have the following values:

$$\begin{aligned}&b_{11} = 1.5, \quad b_{12} = -1.5, \quad b_{21} = -1.5, \quad b_{22} = 1,\\ &d_{11} = 1, \quad d_{12} = 0.5, \quad d_{21} = -1, \quad d_{22} = -1,\\ &I_1 = 10, \quad I_2 = -10, \quad J_1 = -8, \quad J_2 = 8,\\ &\tau_{11} = 10, \quad \tau_{12} = 5, \quad \tau_{21} = 8, \quad \tau_{22} = 7,\\ &\sigma_{11} = 6, \quad \sigma_{12} = 7, \quad \sigma_{21} = 9, \quad \sigma_{22} = 10\end{aligned} \tag{4.2}$$

and the delay kernels $K_{ij}(\cdot),\ H_{ij}(\cdot)$ are assumed to be in the following forms:

$$\begin{aligned}&K_{11}(s) = \frac{0.5\mathrm{e}^{-0.5s}}{1-\mathrm{e}^{-0.5\tau_{11}}}, \qquad K_{12}(s) = \frac{2\mathrm{e}^{-2s}}{1-\mathrm{e}^{-2\tau_{12}}},\\ &K_{21}(s) = \frac{1.5\mathrm{e}^{-1.5s}}{1-\mathrm{e}^{-1.5\tau_{21}}}, \qquad K_{22}(s) = \frac{2.5\mathrm{e}^{-2.5s}}{1-\mathrm{e}^{-2.5\tau_{22}}},\\ &H_{11}(r) = \frac{3\mathrm{e}^{-3r}}{1-\mathrm{e}^{-3\sigma_{11}}}, \qquad H_{12}(r) = \frac{4\mathrm{e}^{-4r}}{1-\mathrm{e}^{-4\sigma_{12}}},\\ &H_{21}(r) = \frac{2.5\mathrm{e}^{-2.5r}}{1-\mathrm{e}^{-2.5\sigma_{21}}}, \qquad H_{22}(r) = \frac{2\mathrm{e}^{-2r}}{1-\mathrm{e}^{-2\sigma_{22}}}\end{aligned} \tag{4.3}$$

for $s \in [0, \tau_{ij}],\ r \in [0, \sigma_{ij}]$. Notice that the kernels $K_{ij}(\cdot),\ H_{ij}(\cdot)$ denote particular forms of (1.5) and one can verify that

$$\int_0^{\tau_{ij}} K_{ij}(s)\,\mathrm{d}s = 1, \qquad \int_0^{\sigma_{ij}} H_{ij}(s)\,\mathrm{d}s = 1$$

for all $i, j = 1,2$.

The associated discrete-time analogue of the network (4.1) is given by

$$\begin{aligned} x_i(n+1) &= \mathrm{e}^{-a_i h} x_i(n) + \theta_i(h) \sum_{j=1}^{2} b_{ij} \tanh\left(\sum_{p=1}^{\kappa_{ij}} \mathscr{K}_{ij}(p) y_j(n-p) \right) + \theta_i(h) I_i, \\ y_i(n+1) &= \mathrm{e}^{-c_i h} y_i(n) + \phi_i(h) \sum_{j=1}^{2} d_{ij} \tanh\left(\sum_{p=1}^{\gamma_{ij}} \mathscr{H}_{ij}(p) x_j(n-p) \right) + \phi_i(h) J_i \end{aligned} \tag{4.4}$$

for $i=1,2,\ n \in \{0,1,2,\ldots\}$ where $\kappa_{ij}=[\tau_{ij}/h],\ \gamma_{ij}=[\sigma_{ij}/h],\ \theta_i(h)=(1-\mathrm{e}^{-a_i h})/a_1,\ \phi_i(h)=(1-\mathrm{e}^{-c_i h})/c_i$ and the analogue kernels $\mathscr{K}_{ij}(\cdot),\ \mathscr{H}_{ij}(\cdot)$ are given by

$$\begin{aligned} &\mathscr{K}_{11}(p) = \frac{1-\mathrm{e}^{-0.5h}}{\mathrm{e}^{-0.5h}} \frac{\mathrm{e}^{-0.5ph}}{1-\mathrm{e}^{-0.5\tau_{11}}}, \qquad \mathscr{K}_{12}(p) = \frac{1-\mathrm{e}^{-2h}}{\mathrm{e}^{-2h}} \frac{\mathrm{e}^{-2ph}}{1-\mathrm{e}^{-2\tau_{12}}}, \\ &\mathscr{K}_{21}(p) = \frac{1-\mathrm{e}^{-1.5h}}{\mathrm{e}^{-1.5h}} \frac{\mathrm{e}^{-1.5ph}}{1-\mathrm{e}^{-1.5\tau_{21}}}, \qquad \mathscr{K}_{22}(p) = \frac{1-\mathrm{e}^{-2.5h}}{\mathrm{e}^{-2.5h}} \frac{\mathrm{e}^{-2.5ph}}{1-\mathrm{e}^{-2.5\tau_{22}}}, \\ &\mathscr{H}_{11}(q) = \frac{1-\mathrm{e}^{-3h}}{\mathrm{e}^{-3h}} \frac{\mathrm{e}^{-3qh}}{1-\mathrm{e}^{-3\sigma_{11}}}, \qquad \mathscr{H}_{12}(q) = \frac{1-\mathrm{e}^{-4h}}{\mathrm{e}^{-4h}} \frac{\mathrm{e}^{-4qh}}{1-\mathrm{e}^{-4\sigma_{12}}}, \\ &\mathscr{H}_{21}(q) = \frac{1-\mathrm{e}^{-2.5h}}{\mathrm{e}^{-2.5h}} \frac{\mathrm{e}^{-2.5qh}}{1-\mathrm{e}^{-2.5\sigma_{21}}}, \qquad \mathscr{H}_{22}(q) = \frac{1-\mathrm{e}^{-2h}}{\mathrm{e}^{-2h}} \frac{\mathrm{e}^{-2qh}}{1-\mathrm{e}^{-2\sigma_{22}}} \end{aligned} \tag{4.5}$$

for $p \in \{1,2,\ldots,\kappa_{ij}\},\ q \in \{1,2,\ldots,\gamma_{ij}\}$. We note that the analogue kernels $\mathscr{K}_{ij}(\cdot),\ \mathscr{H}_{ij}(\cdot)$ satisfy (2.8) and in particular,

$$\sum_{p=1}^{\kappa_{ij}} \mathscr{K}_{ij}(p) \approx \int_0^{\tau_{ij}} K_{ij}(s)\,\mathrm{d}s = 1, \qquad \sum_{p=1}^{\gamma_{ij}} \mathscr{H}_{ij}(p) \approx \int_0^{\sigma_{ij}} H_{ij}(s)\,\mathrm{d}s = 1.$$

To illustrate the convergence dynamics of the continuous-time network (4.1), we consider three neural states

$$\begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{y}(t) \end{pmatrix}, \begin{pmatrix} \boldsymbol{u}(t) \\ \boldsymbol{v}(t) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \boldsymbol{w}(t) \\ \boldsymbol{z}(t) \end{pmatrix}$$

whose initial values are provided by

$$\begin{pmatrix} \boldsymbol{x}(s) \\ \boldsymbol{y}(s) \end{pmatrix} = \begin{pmatrix} x_1(s) \\ x_2(s) \\ y_1(s) \\ y_2(s) \end{pmatrix} = \begin{pmatrix} 10 + 10\mathrm{e}^{0.5s} \\ 5 + 10\sin(0.5\pi s) \\ 20 + 5\sin(\pi s) \\ 2s + 30 \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{u}(s) \\ \boldsymbol{v}(s) \end{pmatrix} = \begin{pmatrix} u_1(s) \\ u_2(s) \\ v_1(s) \\ v_2(s) \end{pmatrix} = \begin{pmatrix} -10s + \cos(2\pi s) \\ -15 \\ 6 - 30\mathrm{e}^{0.5s} \\ -20 \end{pmatrix},$$

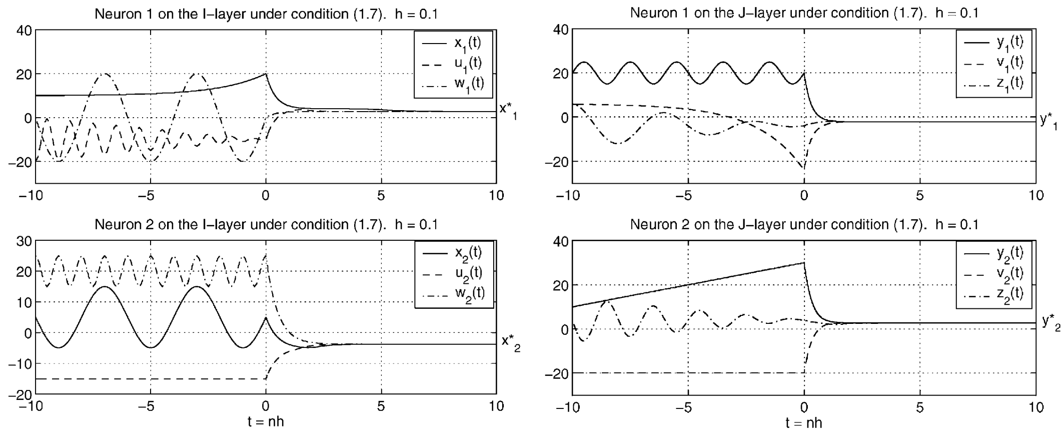

Fig. 1. Exponential convergence of the continuous-time network (4.1) under the stability condition (1.7). The computer simulations are provided by the discrete-time analogue (4.4) at step-size $h = 0.1$.

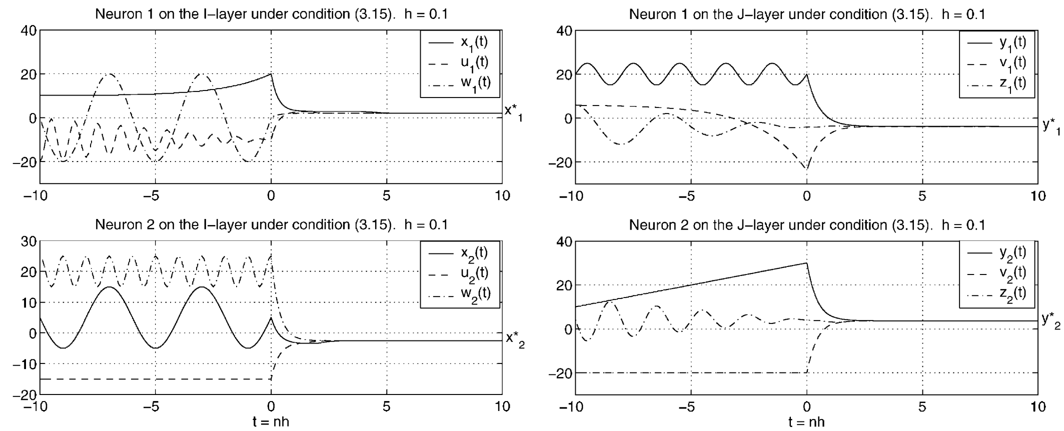

Fig. 2. Exponential convergence of the continuous-time network (4.1) under the stability condition (3.15). The computer simulations are provided by the discrete-time analogue (4.4) at step-size $h = 0.1$.

$$\begin{pmatrix} \boldsymbol{w}(s) \\ \boldsymbol{z}(s) \end{pmatrix} = \begin{pmatrix} w_1(s) \\ w_2(s) \\ z_1(s) \\ z_2(s) \end{pmatrix} = \begin{pmatrix} 20\sin(0.5\pi s) \\ 20 + 5\cos(2\pi s) \\ -4 + s\cos(0.5\pi s) \\ 4 + s\sin(\pi s) \end{pmatrix}$$

for $s \in [-10, 0]$. In Figs. 1 and 2, we use the discrete-time analogue (4.4) at step-size $h = 0.1$ to simulate the convergence dynamics of the continuous-time network (4.1) under the stability condition

(1.7) and (3.15), respectively. It is shown by Fig. 1 that the neural states converge exponentially to the unique equilibrium

$$\begin{pmatrix} \boldsymbol{x}^* \\ \boldsymbol{y}^* \end{pmatrix} = \begin{pmatrix} x_1^* \\ x_2^* \\ y_1^* \\ y_2^* \end{pmatrix} \approx \begin{pmatrix} 2.8217 \\ -3.7749 \\ -2.1477 \\ 2.6699 \end{pmatrix}, \tag{4.6}$$

where the synaptic weights $b_{ij},\ d_{ij}$ are provided by (4.2) and dissipative coefficients $a_i,\ c_i$ are chosen as

$$a_1 = 2.5, \quad a_2 = 2, \quad c_1 = 3.5, \quad c_2 = 3. \tag{4.7}$$

Notice that the stability condition (1.7) is satisfied while the stability condition (3.15) is violated. In Fig. 2, the three neural states, under the stability condition (3.15), converge exponentially to the unique equilibrium

$$\begin{pmatrix} \boldsymbol{x}^* \\ \boldsymbol{y}^* \end{pmatrix} = \begin{pmatrix} x_1^* \\ x_2^* \\ y_1^* \\ y_2^* \end{pmatrix} \approx \begin{pmatrix} 2.001 \\ -2.5 \\ -3.7797 \\ 3.6491 \end{pmatrix}, \tag{4.8}$$

where in this case the dissipative coefficients are chosen as

$$a_1 = 3.5, \quad a_2 = 3, \quad c_1 = 2, \quad c_2 = 2.2 \tag{4.9}$$

and the synaptic weights $b_{ij},\ d_{ij}$ are provided by (4.2); we note that the stability condition (1.7) is violated.

We proceed to demonstrate the capabilities of the discrete-time analogue (4.4) as a numerical algorithm in simulating the continuous-time network (4.1) while preserving the dynamical characteristics of (4.1). In the following exercises, we compare our discrete-time analogue (4.4) with the Euler-type discrete-time model in simulating the continuous-time network (4.1); the Euler-type discrete-time model is given by

$$\begin{aligned} x_i(n+1) &= (1 - a_i h)x_i(n) + h\sum_{j=1}^{2} b_{ij} \tanh\left(\sum_{p=1}^{\kappa_{ij}} \mathscr{K}_{ij}(p) y_j(n-p)\right) + hI_i, \\ y_i(n+1) &= (1 - c_i h)y_i(n) + h\sum_{j=1}^{2} d_{ij} \tanh\left(\sum_{p=1}^{\gamma_{ij}} \mathscr{H}_{ij}(p) x_j(n-p)\right) + hJ_i \end{aligned} \tag{4.10}$$

for $i = 1, 2,\ n \in \{0, 1, 2, \ldots\}$. We consider the convergence dynamics of the continuous-time network (4.1) under the stability condition (1.7), namely, the parameters $a_i,\ c_i,\ b_{ij},\ d_{ij}$ are provided by (4.2) and (4.7). It is shown by Fig. 3 that both discrete-time models (4.4) and (4.10) at step-size $h = 0.1$ preserve the exponential convergence of the continuous-time network (4.1). As the step-size h is increased to $h = 1$ while $a_i,\ c_i,\ b_{ij},\ d_{ij}$ are fixed, we report from Fig. 3 that the Euler-type model (4.10) produces divergent solutions whereas our discrete-time analogue (4.4) persists in preserving the exponential convergence of the continuous-time network (4.1).

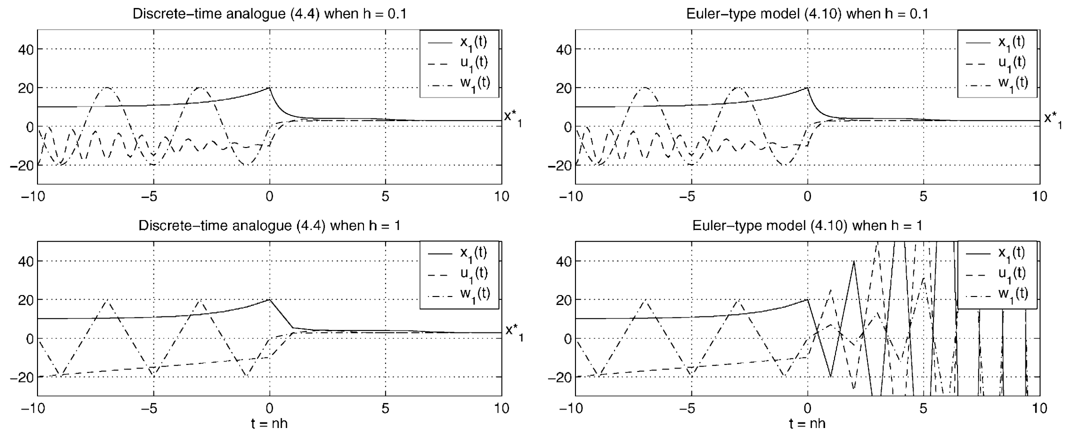

Fig. 3. We compare the discrete-time analogue (4.4) and the Euler-type model (4.10) in numerically simulating the continuous-time network (4.1) at step-sizes $h = 0.1$ and $h = 1$. The Euler-type model (4.10) at $h = 1$ produces divergent solutions; our discrete-time analogue (4.4) preserves the convergence dynamics of the continuous-time network (4.1) at both step-sizes.

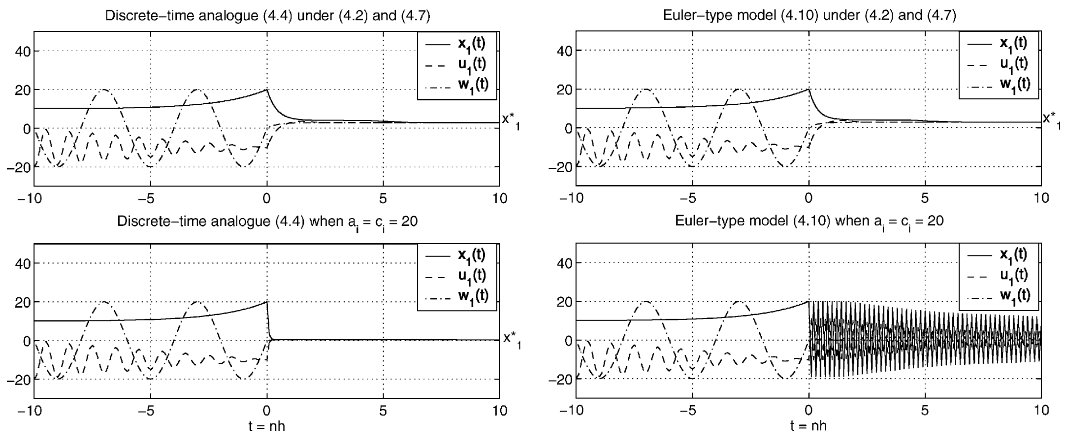

Fig. 4. We compare the computer simulations of the continuous-time network (4.1) produced by the discrete-time analogue (4.4) and the Euler-type model (4.10) at step-size $h=0.1$. The dissipative coefficients a_i, c_i are increased while the synaptic weights b_{ij}, d_{ij} are fixed as in (4.2). Our discrete-time analogue (4.4) produces the desired convergence dynamics of (4.1); the Euler-type model produces spurious oscillatory solutions when $a_i = c_i = 20$.

In Fig. 4, we fix the step-size h at $h = 0.1$ and we fix the synaptic weights b_{ij}, d_{ij} as in (4.2); we increase the dissipative coefficients a_i, c_i to $a_i = c_i = 20$ for $i = 1, 2$. We report from Fig. 4 that the Euler-type model (4.10) produces spurious dynamics in the form of oscillatory solutions whereas our discrete-time analogue (4.4) remains reliable in preserving the exponential convergence of the continuous-time network (4.1).

5. Concluding remarks

We have formulated a discrete-time analogue of a system of integrodifferential equations modelling a bidirectional neural network and we have established the convergence characteristics of the discrete-time analogue. Under identical sufficient condition regardless of the choice of the discretization step-size, the discrete-time analogue inherits the unique equilibrium of the continuous-time network, and moreover the discrete-time analogue converges exponentially to the unique equilibrium. The global exponential stability of the discrete-time analogue simply indicates an improvement over the global asymptotic stability of the continuous-time network established by Gopalsamy and He [4]. In addition, we have also obtained another sufficient condition for the global exponential stability of the discrete-time analogue by employing discrete Halanay-type inequalities; the result obtained indirectly establishes (by adopting analogous proof of Theorem 3.2) the global exponential stability of the continuous-time neural network.

The analytical study of the discrete-time analogue was then extended to actual implementation of the discrete-time analogue in numerically simulating the continuous-time network with delays distributed over finite intervals. In the examples provided in the previous section, we saw some of the wonderful features of the discrete-time analogue in providing the computer simulations of the continuous-time network. Among them is the superiority of the discrete-time analogue over the conventional Euler method in simulating the continuous-time network while preserving the exponential convergence of the continuous-time network. These wonderful features possessed by the discrete-time analogue are partly due to the fact that the discretization step-size does not appear in the stability conditions.

We end this article with the following: Does there exist another discrete-time analogue which can preserve the exponential convergence of the continuous-time network (1.1) without restricting the discretization step-size? The answer is yes. The interested reader is referred to [13–15] for the formulation of such discrete-time analogue.

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